UPPSALA UNIVERSITET
Matematiska institutionen
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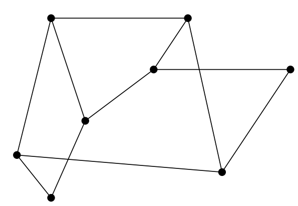
PROV I MATEMATIK
Grafteori / Graph theory
2015-08-17

Time: 14–19. Only pen(cil), eraser and paper may be used (paper is supplied).
In order to pass the course it is necessary to supply a correct proof, explanation or statement (according to the nature of the question) in every assignment that follows. In addition to this exam one also has to solve home work assignments. This will determine the grade, according to the rules explained in the beginning of the course and available on the course page on Studentportalen.

1. (a) Define the notion of *matching* of A in a graph $G = (V, E)$, where $A \subseteq V$.
(b) Let $E' \subseteq E$ where $G = (V, E)$ is a graph. Define the notion of *vertex cover* of E' *and* then state *König's theorem* about matchings.
(c) Let $G = (V, E)$ denote the graph illustrated below. What is the minimal cardinality of a vertex cover of E?
(d) Use König's theorem to determine the maximal cardinality of a matching in G.
(e) Answer the following question by *only* using your answer from part (d) and the number of vertices in G: Does G have a complete matching, that is, a matching such that every vertex is matched with another vertex?



2. (a) Define the notions *vertex colouring* and *chromatic number*.
(b) Describe a "greedy algorithm" for vertex colouring which, given any graph $G = (V, E)$, produces a colouring of G which uses at most $\Delta(G) + 1$ colours, where
$\Delta(G) = \max\{d(v) : v \in V\}$ and $d(v)$ is the degree of v.
(c) Show, by giving an example, that for different orderings of the vertex set of a graph G this algorithm may produce different colourings using different numbers of colours.

3. (a) Define what it means for a graph to be a *tree* and what it means for a graph to be a *forest*.
(b) Explain the proof of Euler's formula for connected plane graphs $G = (V, E)$:

$$n - m + l = 2, \quad \text{where } n = |V|,\ m = |E| \text{ and } l \text{ is the number of faces of } G.$$

Do this in an intutitive way, without notions from topology. Start by explaining why the formula holds if G is a tree. Then apply induction on the number of edges of G.
(c) Use Euler's formula to show that if G is a planar graph with $n \geq 3$ vertices, then G has at most $3n - 6$ edges.

4. (a) State the "maximum-flow minimum-cut" theorem for network flows and define all notions appearing in that theorem.
(b) Explain how the "maximum-flow minimum-cut" theorem can be used to find a maximal matching for any bipartite graph. This must include an explanation for why the method necessarily produces a maximum matching. Start by telling how a bipartite graph can be converted into a network which can be used to solve the problem.

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5. Let $G = (V, E)$ be a graph and let P_{ij} denote the following property:

> Whenever $X, Y \subseteq V$, $|X| = i$, $|Y| = j$ and $X \cap Y = \emptyset$, then there is $v \in V \setminus (X \cup Y)$ such that v is adjacent to every member of X and v is nonadjacent to every member of Y.

(a) Let $i, j \geq 0$ be fixed integers. Show that the proportion of all graphs $G = (V, E)$ with $V = \{1, \ldots, n\}$ that satisfy P_{ij} approaches 1 as $n \to \infty$.
(b) Suppose that $G = (V, E)$ is a graph such that $|V| > 0$ and G satisfies P_{ij} for every choice of $i, j \geq 0$. Explain why G must be infinite.

Good luck!